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CONJUGATE SYSTEMS WITH EQUAL POINT INVARIANTS.

BY LUTHER PFAHLER EISENHART.

When the developables of a rectilinear congruence meet a surface S in a conjugate system of curves, the conjugate system and the congruence are said to be *conjugate* to one another. When the conjugate system on S is parametric, its point coördinates satisfy a partial differential equation of the Laplace type. Each solution μ of the adjoint differential equation determines a congruence conjugate to the given system. Each solution θ of the point equation of S determines a new conjugate system S_1 conjugate to this congruence. We say that S_1 is obtained from S by a *transformation* $T(\mu, \theta)$.

The interrelations of S and S_1 with their Laplace transforms S^{-1}, S' and S_1^{-1}, S_1' are studied. The tangent planes to S osculate one family of curves on S^{-1} and the other family on S'. Accordingly there is a pencil of conics each of which is tangent to these curves at corresponding points. There exist similar pencils of conics in the tangent planes to S_1. We show that the pencils in corresponding planes cut out on the intersection of these planes the same involution only in case the point equation of S has equal invariants and also that of S_1. We have thus a characteristic property of the transformations K.* In this case the conics of the two pencils are paired, so that those of a pair meet and determine a pencil of quadrics. Certain properties of these pencils of quadrics are determined. It is shown also that each congruence conjugate to a conjugate system with equal point invariants leads at once to a transformation K. This result is useful, in particular, in the determination of surfaces Ω.† Throughout the paper homogeneous coördinates are used, and the results are projective.

1. Transformations $T(\mu, \theta)$.

The necessary and sufficient condition that four functions x, y, z, w be the homogeneous point coördinates of a surface S referred to a conjugate system of parameters u and v is that these functions satisfy an equation of the form

$$(1) \qquad \frac{\partial^2\theta}{\partial u \partial v} + a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v} + c\theta = 0,$$

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* Transactions of the American Mathematical Society, vol. 15 (1914), pp. 397–430.

† Transactions of the American Mathematical Society, vol. 16 (1915), pp. 275–310.

where a, b, c are functions of u and v. We refer to this equation as the point equation of the conjugate system.

Darboux* has shown that when a solution μ of the equation adjoint to (1), namely

$$\frac{\partial^2\mu}{\partial u\partial v} - a\frac{\partial\mu}{\partial u} - b\frac{\partial\mu}{\partial v} + \left(c - \frac{\partial a}{\partial u} - \frac{\partial b}{\partial v}\right)\mu = 0, \tag{2}$$

is known, a congruence conjugate to the given conjugate system is given by quadratures. In fact, the coördinates X_1, Y_1, Z_1, W_1; X_2, Y_2, Z_2, W_2, of the focal points F_1 and F_2 of this congruence are given by equations of the form

$$\tag{3} \begin{aligned} X_1 &= \int\left[\mu\left(\frac{\partial x}{\partial u} + bx\right)du + x\left(\frac{\partial\mu}{\partial v} - a\mu\right)dv\right], \\ X_2 &= \int\left[x\left(\frac{\partial\mu}{\partial u} - b\mu\right)du + \mu\left(\frac{\partial x}{\partial v} + ax\right)dv\right]. \end{aligned}$$

Each solution θ of (1), linearly independent of x, y, z, w, leads by quadratures to a conjugate system conjugate to the above congruence. In fact, if μ is the function appearing in (3) and θ is such a solution of (1), the function σ given by

$$\sigma = \int\left[\mu\left(\frac{\partial\theta}{\partial u} + b\theta\right)du + \theta\left(\frac{\partial\mu}{\partial v} - a\mu\right)dv\right] \tag{4}$$

is a solution of the point equation of (F_1), the locus of F_1, and is linearly independent of X_1, Y_1, Z_1, W_1. Hence by the theorem of Levy† the surface S_1, whose coördinates x_1, y_1, z_1, w_1 are given by equations of the form

$$x_1 = X_1 - \frac{\sigma}{\frac{\partial\sigma}{\partial v}}\frac{\partial X_1}{\partial v} = X_1 - \frac{\sigma x}{\theta}, \tag{5}$$

is conjugate to the congruence whose focal surfaces are (F_1) and (F_2). We say that the new conjugate system is obtained from the original one by a *transformation* $T(\mu, \theta)$.

If μ is held fixed, and θ is any solution of (1), we get all the conjugate systems conjugate to this congruence. Hence equations (3), (4) and (5) define the general transformation $T(\mu, \theta)$.

From (5) we find at once

$$\frac{\partial x_1}{\partial u} = (\mu\theta - \sigma)\frac{\partial}{\partial u}\left(\frac{x}{\theta}\right), \qquad \frac{\partial x_1}{\partial v} = -\sigma\frac{\partial}{\partial v}\left(\frac{x}{\theta}\right). \tag{6}$$

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* Leçons, vol 2, p. 225.

† Journal de l'Ecole Polytechnique, cahier 56 (1886), p. 63; also Darboux, Leçons, vol. 2, p. 222.

From these equations we see that the points $T_1(\xi_1, \eta_1, \zeta_1, \omega_1)$, $T_2(\xi_2, \eta_2, \zeta_2, \omega_2)$, whose coördinates are given by equations of the form

$$\xi_1 = \theta\frac{\partial x}{\partial u} - x\frac{\partial \theta}{\partial u}, \qquad \xi_2 = \theta\frac{\partial x}{\partial v} - x\frac{\partial \theta}{\partial v}, \tag{7}$$

are the points of intersection of the tangents to the curves v = const. and the curves u = const. respectively on S and S_1.

By differentiation we find

$$\begin{aligned} \frac{\partial \xi_1}{\partial v} &= \left(\frac{\partial \log \theta}{\partial v} - a\right)\xi_1 - \left(\frac{\partial \log \theta}{\partial u} + b\right)\xi_2, \\ \frac{\partial \xi_2}{\partial u} &= -\left(\frac{\partial \log \theta}{\partial v} + a\right)\xi_1 + \left(\frac{\partial \log \theta}{\partial u} - b\right)\xi_2. \end{aligned} \tag{8}$$

Consequently T_1 and T_2 are the focal points of the congruence of lines T_1T_2.

The equations of S_1 can be given another form, if we look upon the lines of the given congruence as tangent to the focal surface (F_2) also. Evidently the function τ, given by

$$\tau = \int\left[\theta\left(\frac{\partial \mu}{\partial u} - b\mu\right)du + \mu\left(\frac{\partial \theta}{\partial v} + a\theta\right)dv\right], \tag{4'}$$

is a solution of the point equation of (F_2). The equations of S_1 are of the form

$$x_1 = -X_2 + \frac{\tau x}{\theta}, \tag{5'}$$

and we have also

$$\frac{\partial x_1}{\partial u} = \tau\frac{\partial}{\partial u}\left(\frac{x}{\theta}\right), \qquad \frac{\partial x_1}{\partial v} = -(\mu\theta - \tau)\frac{\partial}{\partial v}\left(\frac{x}{\theta}\right). \tag{6'}$$

It is readily shown that these equations are consistent with (5) and (6), if we take

$$\tau = \mu\theta - \sigma, \tag{9}$$

which is consistent also with (4) and (4′).

2. Transformations of Laplace of S and S_1.

If lines be drawn tangent to the curves v = const. on S, we get a congruence one of whose focal surfaces is S, and the other focal surface S^{-1}, called a *Laplace transform* of S, is defined by equations of the form

$$x^{-1} = \frac{\partial x}{\partial u} + bx, \tag{10}$$

and similar expressions for the other coördinates y^{-1}, z^{-1}, w^{-1} of the point M^{-1}.* In like manner the coördinates x', y', z', w' of M', the corresponding point on the other Laplace transform, obtained by drawing the tangents to the curves u = const. on S, are of the form

$$x' = \frac{\partial x}{\partial v} + ax. \tag{11}$$

From (6) we derive the point equation of S_1, namely

$$\frac{\partial^2 \theta_1}{\partial u \partial v} + a_1 \frac{\partial \theta_1}{\partial u} + b_1 \frac{\partial \theta_1}{\partial v} = 0, \tag{12}$$

where

$$a_1 = \frac{\sigma\left(a + \dfrac{\partial \log \theta}{\partial v}\right)}{\sigma - \mu\theta}, \qquad b_1 = \frac{\sigma - \mu\theta}{\sigma}\left(b + \frac{\partial \log \theta}{\partial u}\right). \tag{12'}$$

The equations of the Laplace transforms S_1^{-1} and S_1' of S_1 are analogous to (10) and (11). By means of (5), (5'), (6), (6'), (9) and (12') these are reducible to

$$\begin{aligned} x_1^{-1} &= \frac{\partial x_1}{\partial u} + b_1 x_1 = b_1 X_1 + \frac{\mu\theta - \sigma}{\theta} x^{-1}, \\ x_1' &= \frac{\partial x_1}{\partial v} + a_1 x_1 = -a_1 X_2 + \frac{\tau - \mu\theta}{\theta} x'. \end{aligned} \tag{13}$$

From (3), (10) and (11) we have

$$\frac{\partial X_1}{\partial u} = \mu x^{-1}, \qquad \frac{\partial X_2}{\partial v} = \mu x'. \tag{14}$$

Comparing these results, we have

THEOREM 1. *Corresponding points on S^{-1} and S_1^{-1} lie on the corresponding tangent to the curve v = const. on (F_1); corresponding points on S' and S_1' lie on the corresponding tangent to the curve u = const. on (F_2).*

3. Pencils of conics in the tangent planes to S and S_1.

In order to study certain configurations associated with transformations $T(\mu, \theta)$, we take for tetrahedron of reference $MM_1T_1T_2$, where M and M_1 are corresponding points on S and S_1, and T_1 and T_2 are given by (7). If we take

$M(0, 1, 0, 1)$, $M_1(0, 0, 0, 1)$, $T_1(1, 0, 0, 0)$, $T_2(0, 0, 1, 0)$,

we have from (6) and (7)

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* Darboux, Leçons, vol. 2, p. 19.

$$\frac{\partial x}{\partial u} = \frac{1}{\theta}, \qquad \frac{\partial y}{\partial u} = \frac{\partial \log \theta}{\partial u}, \qquad \frac{\partial z}{\partial u} = \frac{\partial w}{\partial u} = 0,$$

$$\frac{\partial x}{\partial v} = 0, \qquad \frac{\partial y}{\partial v} = \frac{\partial \log \theta}{\partial v}, \qquad \frac{\partial z}{\partial v} = \frac{1}{\theta}, \qquad \frac{\partial w}{\partial v} = 0.$$

Hence from (10) and (11) follow

$$x^{-1} = \frac{1}{\theta}, \qquad y^{-1} = \frac{\partial \log \theta}{\partial u} + b, \qquad z^{-1} = w^{-1} = 0,$$

$$x' = 0, \qquad y' = \frac{\partial \log \theta}{\partial v} + a, \qquad z' = \frac{1}{\theta}, \qquad w' = 0.$$

We have denoted by M^{-1} and M' the points with these coördinates.

The pencil of conics in the tangent plane to S at M tangent to MM^{-1} and MM' at M^{-1} and M' is given by

$$(15) \qquad 2\lambda xz = \left[\left(\frac{\partial \log \theta}{\partial u} + b\right)x - \frac{y}{\theta} + \left(\frac{\partial \log \theta}{\partial v} + a\right)z\right]^2.$$

From the theory of congruences it follows that the tangent plane at M is the osculating plane of the curve $u =$ const. on S^{-1} at M^{-1}. We seek λ so that the above equation shall give a conic osculating $u =$ const. at this point. The necessary and sufficient condition is that the above equation shall be satisfied to within terms of the second order in dv by expressions of the form

$$x^{-1} + \frac{\partial x^{-1}}{\partial v} dv + \frac{1}{2}\frac{\partial^2 x^{-1}}{\partial v^2} dv^2.$$

From (10) we have in general

$$\frac{\partial x^{-1}}{\partial v} = -a\frac{\partial x}{\partial u} + \left(\frac{\partial b}{\partial v} - c\right)x, \qquad \frac{\partial^2 x^{-1}}{\partial v^2} = A\frac{\partial x}{\partial u} + k\frac{\partial x}{\partial v} + Bx,$$

where

$$(16) \quad A = a^2 - \frac{\partial a}{\partial v}, \qquad k = \frac{\partial b}{\partial v} + ab - c, \qquad B = ac + \frac{\partial^2 b}{\partial v^2} - \frac{\partial c}{\partial v}.$$

For the present choice of tetrahedron we have

$$\frac{\partial x^{-1}}{\partial v} = -\frac{a}{\theta}, \qquad \frac{\partial y^{-1}}{\partial v} = -a\frac{\partial \log \theta}{\partial u} + \frac{\partial b}{\partial v} - c, \qquad \frac{\partial z^{-1}}{\partial v} = \frac{\partial w^{-1}}{\partial v} = 0,$$

$$\frac{\partial^2 x^{-1}}{\partial v^2} = \frac{A}{\theta}, \qquad \frac{\partial^2 y^{-1}}{\partial v^2} = A\frac{\partial \log \theta}{\partial u} + k\frac{\partial \log \theta}{\partial v} + B,$$

$$\frac{\partial^2 z^{-1}}{\partial v^2} = \frac{k}{\theta}, \qquad \frac{\partial^2 w^{-1}}{\partial v^2} = 0.$$

Making use of these values, we find that in the pencil of conics (15) the osculating conic of $u =$ const. is given by $\lambda = k$.

Proceeding in like manner, we find that, when

$$\lambda = h = \frac{\partial a}{\partial u} + ab - c, \tag{17}$$

the corresponding conic of the pencil (15) osculates the curve v = const. at M'.

The functions h and k, defined by (17) and (16) are the invariants of equation (1). A consequence of the above results is the theorem of Koenigs:

THEOREM 2. *A necessary and sufficient condition that there exist a conic of the pencil* (15) *osculating the curves* u = *const. on* S^{-1} *and* v = *const. on* S' *is that the invariants of equation* (1) *be equal.*

By the same sort of processes and with the aid of the foregoing formulas we can show that the conics in the tangent plane to S_1 at M_1 and tangent at M_1^{-1} and M_1' to the lines $M_1M_1^{-1}$ and M_1M_1' respectively are defined by

$$2xz = \lambda_1 \left[\frac{x}{\sigma}\left(b + \frac{\partial \log \theta}{\partial u}\right) - \frac{z}{\tau}\left(a + \frac{\partial \log \theta}{\partial v}\right) + \frac{w}{\theta^2} \right]^2. \tag{18}$$

When $-\sigma\tau/\lambda_1$ is equal to k_1 or h_1, the invariants of (12), the corresponding conics osculate the curves u = const. on S_1^{-1} or v = const. on S_1'.

Each of the pencils of conics (15) and (18) determines an involution on the line T_1T_2 ($y = w = 0$), and the points T_1, T_2 are a conjugate pair in each involution. In order that these involutions be identical, it is necessary and sufficient that they have another conjugate pair in common. From equations (15) and (18) it is seen that a necessary condition for this is

$$\sigma^2 = \tau^2.$$

Since $\mu \neq 0$, $\theta \neq 0$, we have from (9)

$$\sigma = \tau = \tfrac{1}{2}\mu\theta.$$

In order that this value may satisfy (4), we must have

$$\frac{\partial}{\partial u} \log \sqrt{\frac{\mu}{\theta}} = b, \qquad \frac{\partial}{\partial v} \log \sqrt{\frac{\mu}{\theta}} = a. \tag{19}$$

The consistency of these equations requires that the invariants h and k of equation (1) shall be equal.

4. Transformations K. Osculating conics.

We consider now the case where $h = k$, and are take equation (1) in the form

$$\frac{\partial^2\theta}{\partial u\partial v}+\frac{\partial \log \sqrt{\rho}}{\partial v}\frac{\partial\theta}{\partial u}+\frac{\partial \log \sqrt{\rho}}{\partial u}\frac{\partial\theta}{\partial v}+c\theta=0. \tag{20}$$

The invariants of this equation have the value

$$h=k=\frac{\partial^2 \log \sqrt{\rho}}{\partial u\partial v}+\frac{\partial \log \sqrt{\rho}}{\partial u}\frac{\partial \log \sqrt{\rho}}{\partial v}-c. \tag{21}$$

By choosing suitably the constant of integration in (19), we have

$$\mu=2\rho\theta,\qquad \sigma=\rho\theta^2.$$

It is readily shown that this value of μ satisfies the equation adjoint to (20)

The corresponding equations (6) are

$$\frac{\partial x_1}{\partial u}=\rho\theta^2\frac{\partial}{\partial u}\left(\frac{x}{\theta}\right),\qquad \frac{\partial x_1}{\partial v}=-\rho\theta^2\frac{\partial}{\partial v}\left(\frac{x}{\theta}\right). \tag{22}$$

The point equation of S_1 is

$$\frac{\partial^2\theta_1}{\partial u\partial v}-\frac{\partial}{\partial v}\log\theta\sqrt{\rho}\frac{\partial\theta_1}{\partial u}-\frac{\partial}{\partial u}\log\theta\sqrt{\rho}\frac{\partial\theta_1}{\partial v}=0, \tag{23}$$

so that its invariants are equal and are given by

$$h_1=k_1=2\frac{\partial \log \sqrt{\rho}\theta}{\partial u}\frac{\partial \log \sqrt{\rho}\theta}{\partial v}-h. \tag{24}$$

From (5) and (5′) it follows that the coördinates of the focal points of the congruence of joins of corresponding points M and M_1 on S and S_1 are of the form

$$X_1=x_1+\rho\theta x,\qquad X_2=-x_1+\rho\theta x. \tag{25}$$

Hence the focal points are harmonic to M and M_1, and we have

THEOREM 3. *When equation* (1) *has equal invariants, to each solution* θ *there corresponds a transformation* $T(2\rho\theta,\theta)$ *such that on* S_1 *the parametric conjugate system has equal invariants, and the focal points of the lines joining corresponding points on* S *and* S_1 *are harmonic to the latter points.*

Accordingly equations (22) define in homogeneous coördinates the transformation K which we have considered at length elsewhere.* In fact, if the homogeneous coördinates are rectangular so that $w=1$, then $c=0$. Now w_1 is given by

$$\frac{\partial w_1}{\partial u}=-\rho\frac{\partial\theta}{\partial u},\qquad \frac{\partial w_1}{\partial v}=\rho\frac{\partial\theta}{\partial v}. \tag{26}$$

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* Transactions, l. c.

Hence the rectangular non-homogeneous coördinates x_1, y_1, z_1 of S_1 are given by

$$(27)\qquad \frac{\partial}{\partial u}(x_1 w_1) = -\rho\left(x\frac{\partial\theta}{\partial u} - \theta\frac{\partial x}{\partial u}\right), \qquad \frac{\partial}{\partial v}(x_1 w_1) = \rho\left(x\frac{\partial\theta}{\partial v} - \theta\frac{\partial x}{\partial v}\right),$$

which are the same as the equations previously obtained.

In the case of a transformation K the equations (15) and (18) are reducible to

$$(28)\qquad 2exz = \left(\frac{\partial \log \theta\sqrt{\rho}}{\partial u}x - \frac{y}{\theta} + \frac{\partial \log \theta\sqrt{\rho}}{\partial v}z\right)^2,$$

$$(29)\qquad 2fxz = \left(\frac{\partial \log \theta\sqrt{\rho}}{\partial u}x - \frac{\partial \log \theta\sqrt{\rho}}{\partial v}z + \rho w\right)^2,$$

where e and f are the parameters of these pencils of conics. The necessary and sufficient condition that a conic of each pencil meet on $y = w = 0$ is that the parameters satisfy the equation

$$(30)\qquad f = e - 2\frac{\partial \log \sqrt{\rho\theta}}{\partial u}\cdot\frac{\partial \log \sqrt{\rho\theta}}{\partial v}.$$

From (17) it follows that when $e = h = k$, the conic osculates the curve $u =$ const. on S^{-1} at M^{-1} and the curve $v =$ const. on S' at M'. We call the conic C_0. We obtain the analogous conic C_{10}, osculating the respective curves on S_1^{-1} and S_1^{1} by putting in (29) $f = -h_1 = -k_1$. In consequence of (24) these values of e and f satisfy (30). Hence the conics C_0 and C_{10} meet on the line $y = w = 0$, and we have

THEOREM 4. *When S and S_1 are two surfaces in the relation of a transformation K, the pencils of conics* (15) *and* (18) *determine the same involution on the line of intersection of the tangent planes to S and S_1; in particular the osculating conics C_0 and C_{10} meet on this line.**

Consider two conics of the pencils (28) and (29) whose parameters satisfy the condition (30). Through these two conics there pass a pencil of quadrics, whose equation is

$$\begin{aligned}\left(\frac{\partial \log \sqrt{\rho\theta}}{\partial u}\right)^2 x^2 + \frac{y^2}{\theta^2} + \left(\frac{\partial \log \sqrt{\rho\theta}}{\partial v}\right)^2 z^2 + \rho^2 w^2 - \frac{2}{\theta}\frac{\partial \log \sqrt{\rho\theta}}{\partial u}xy \\ (31)\qquad + 2\left(\frac{\partial \log \sqrt{\rho\theta}}{\partial u}\frac{\partial \log \sqrt{\rho\theta}}{\partial v} - e\right)xz + 2\rho\frac{\partial \log \sqrt{\rho\theta}}{\partial u}xw \\ - \frac{2}{\theta}\frac{\partial \log \sqrt{\rho\theta}}{\partial v}yz + 2gyw - 2\rho\frac{\partial \log \sqrt{\rho\theta}}{\partial v}zw = 0,\end{aligned}$$

where g is the parameter of the pencil.

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* The last part of this theorem is due to Tzitzeica, Comptes Rendus, 30 Nov., 1908.

These quadrics determine an involution on the line MM_1. The equation of the involution is

$$y^2 + 2g\theta^2 yw + \rho^2\theta^2 w^2 = 0.$$

The double points of the involution are given by

$$(32) \qquad y + gw\theta^2 = 0, \qquad g = \pm\frac{\rho}{\theta}.$$

Comparing this result with (25), we have

THEOREM 5. *Each pencil of quadrics determined by a pair of the conics* (28) *and* (29) *meets the line* MM_1 *in conjugate pairs of an involution whose double points are the focal points of* MM_1 *in the congruence of these lines.*

For each value of e in general two of the quadrics of the pencil (31) are cones. They correspond to the values of g which make the discriminant of (31) vanish. This discriminant equated to zero is equivalent to

$$ef(\rho^2 - g^2\theta^2) = 0.$$

When $e = 0$, the corresponding conic of the pencil (28) is the line $M^{-1}M'$ counted twice. The corresponding conic (29) meets $M^{-1}M'$ in a single point and consequently all the quadrics (31) are cones whose vertices are on this line. Similar results follow when we take $f = 0$. Excluding these two cases, we get equations (32) again. Hence we have

THEOREM 6. *The cones of a pencil of quadrics* (31) *are tangent to the lines* MM_1 *at its focal points, unless all the quadrics of the pencil are cones.*

The coördinates of the vertex of either cone are given by the corresponding determinants of the third order of the discriminant of (31). These are found to be for

$$(33) \qquad \begin{aligned} g\theta + \rho = 0, &\qquad \left[\frac{1}{\theta}\frac{\partial}{\partial v}(\rho\theta^2),\ \frac{\partial \log \sqrt{\rho\theta}}{\partial u}\,\frac{\partial}{\partial v}(\rho\theta^2) - e\rho\theta^2,\ 0,\ -e\theta\right], \\ g\theta - \rho = 0, &\qquad \left[0,\ \frac{\partial \log \sqrt{\rho\theta}}{\partial v}\,\frac{\partial}{\partial u}(\rho\theta^2) - e\rho\theta^2,\ \frac{1}{\theta}\frac{\partial}{\partial u}(\rho\theta^2),\ e\theta\right]. \end{aligned}$$

In § 2 we showed that the lines $M^{-1}M_1^{-1}$ and $M'M_1'$ pass through F_1 and F_2 respectively. We shall show that the vertices (33) lie on these respective lines. In fact, it follows readily from (10), (11) and (25) that the coördinates of these points are of the respective forms.

$$(34) \qquad \frac{\partial}{\partial v}\rho\theta^2 \cdot x^{-1} - e\theta X_1, \qquad \frac{\partial}{\partial u}\rho\theta^2 \cdot x' - e\theta X_2.$$

We consider now the congruences of lines $M^{-1}M_1^{-1}$ and $M'M_1'$. From Theorem 2 we have that F_1 and F_2 are focal points of these lines in the congruences. In order to find the other focal points, we differentiate the expressions (34) respectively with respect to v and u. By means of (10), (11), (21), (22) and (25) the resulting expressions are reducible to

$$\left[\frac{\partial^2}{\partial v^2}(\theta\rho^2) - \frac{\partial}{\partial v}(\rho\theta^2) \quad \frac{\partial}{\partial v}\log\sqrt{\rho}\right]x^{-1} - \frac{\partial}{\partial v}(e\theta)X_1 + \frac{\partial}{\partial v}(\rho\theta^2)(h-e)x,$$

$$\left[\frac{\partial^2}{\partial u^2}(\rho\theta^2) - \frac{\partial}{\partial u}(\rho\theta^2)\frac{\partial}{\partial u}\log\sqrt{\rho}\right]x' - \frac{\partial}{\partial u}(e\theta)X_2 + \frac{\partial}{\partial u}(\rho\theta^2)(k-e)x.$$

From these expressions it follows that when $e = h$, the vertices of the cones are the focal points of these congruences. Hence we have the theorem of Tžitzeica:*

THEOREM 7. *The pencil of quadrics determined by the osculating conics contain two cones whose vertices are focal points of the lines* $M^{-1}M_1^{-1}$ *and* $M'M_1'$.

It is our purpose now to show that every congruence conjugate to a conjugate system with equal point invariants gives a transformation K of the given conjugate system.

Equation (20) may be written

$$\frac{\partial^2}{\partial u \partial v}(\theta\sqrt{\rho}) = \left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial u\partial v} - c\right)\theta\sqrt{\rho},$$

and its adjoint is

$$\frac{\partial^2}{\partial u\partial v}\left(\frac{\mu}{\sqrt{\rho}}\right) = \left(\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial u\partial v} - c\right)\frac{\mu}{\sqrt{\rho}}.$$

Hence the solutions of the given equation and its adjoint may be paired by means of the relation

$$(35) \qquad \mu = 2\rho\theta.$$

Suppose then that we have a solution μ. This determines a congruence conjugate to the given system. If we take for θ the value given by (35), we find that the corresponding conjugate system is given by (22), which shows that the congruence is that of a transformation K.

We consider a single application of this result. Let Σ be a surface referred to its lines of curvature and let x, y, z be its rectangular coördinates. Each solution of its point equation leads to a surface S_0 cut by the developables of the normals to Σ in a conjugate system, which necessarily is conjugate to this congruence. If this conjugate system has equal point invariants, it follows from the above result that the points harmonic of the points of S_0 with respect to the centers of principal curvature of Σ

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* L. c.

determine a second conjugate system with equal point invariants conjugate to the normal congruence. In this case Σ is a so-called *surface* Ω. Hence the necessary and sufficient condition that Σ be a surface Ω is found by expressing the condition that on an S_0 the conjugate system has equal point invariants.*

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* We have derived the analytical condition for surfaces Ω in the Transactions of the American Mathematical Society, vol. 16, pp. 280–283. This condition is more readily obtained by making use of the results of the present paper.